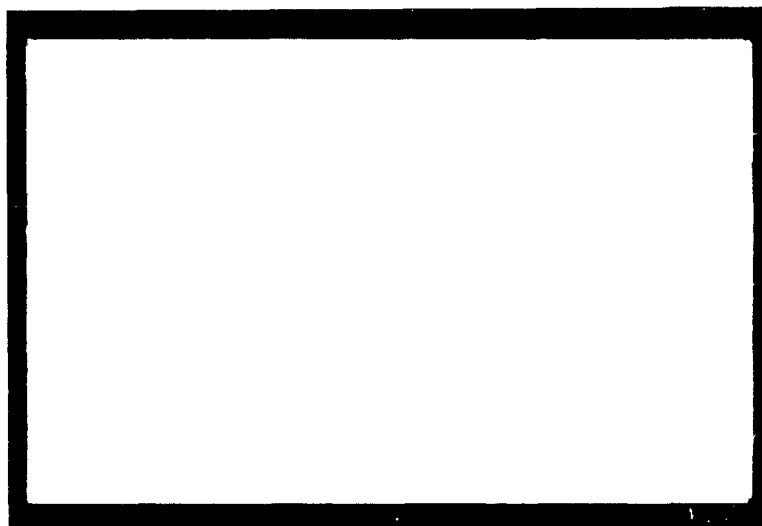
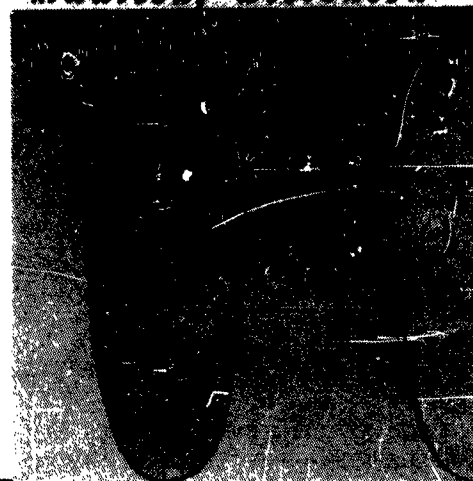


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TIED-DOUBLE-CHANGE-OVER DESIGNS

W. T. Federer and G. F. Atkinson

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ABSTRACT

The construction and analysis for a class of experimental designs denoted as tied-double-change-over designs are presented. These designs are useful for situations wherein the treatments are applied in sequence to an experimental unit and where the effect of a treatment persists for one period after the period in which the treatment was applied; they allow estimation of direct and residual treatment effects. Tied-double-change-over designs are constructed utilizing one, two, ..., $t - 1$ orthogonal latin squares for t treatments. Although the analysis is for r rows and for c columns in general, particular attention is given to the case where $r = tq + 1$ rows and $c = ts$ columns for $sq = k(t - 1)$, for k a positive integer; explicit solutions are obtained for the situations where the first period results are omitted from the analysis and where the first period results are included. A numerical example is used to illustrate the application of the results to experimental data.

TIED-DOUBLE-CHANGE-OVER DESIGNS^{L1}

W. T. Federer^{L2} and G. F. Atkinson^{L3}

I. INTRODUCTION

The experimental design for three treatments in a seven row by six column design constructed to estimate direct and residual effects was described on page 454 of Federer [1955]. The analysis of variance and estimators for effects for this design were presented by Federer and Ferris [1956]. They denoted this design as a tied-double-change-over design to distinguish it from the double change-over or similar designs as discussed by Cochran et al. [1941], Williams [1949, 1950], Patterson [1950, 1951, 1952], Lucas [1951], Ferris [1957], Patterson and Lucas [1959], and Sheehe and Bross [1961]. The nomenclature utilized is analogous to that used by Pearce [1953] for his tied-latin square designs wherein a sequence of treatments is not applied to the same experimental unit.

In this paper the general methods of constructing tied-double-change-over designs involving t treatments, r rows (periods), and c columns (sequences) are presented. The estimators for effects, their variances, and computing formulae for the sums of squares in the analysis of variance are developed.

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Particular attention is directed to two special cases:

$$i) \ t = 3, \ r = 3q + 1, \ c = 3s$$

$$ii) \ t = 4, \ r = 4q + 1, \ c = 4s \ .$$

A numerical example for $t = 3$, $q = 2$, $s = 2$ is presented to illustrate the computing procedure. The general case for r rows and c columns is discussed to some extent.

II CONSTRUCTION

The construction of tied-double-change-over designs is described via examples. There are basically two types of construction, one involving the use of $t - 1$ orthogonal latin squares and the other involving one square for t even, two squares for t odd, or a subset of the orthogonal latin squares. The nearer the design is to a balanced arrangement for direct and residual effects of treatments the more nearly equal will be the variances of differences between direct and residual treatment effects. The relative variances for direct and residual effects approach equality as the number of rows, r , increases.

The residual effects considered in this paper are of a specific kind, viz., those that exert an influence or effect only on the observation in the period (row) immediately following the period in which the treatment was applied. If the residual effect of a treatment lasts longer than one period this must be considered in constructing designs to measure residual effects in the successive periods following application of the treatment. Williams [1949] and Patterson [1952], have presented results for designs of this type. Also, if a direct effect by residual effect interaction exists this must also be taken into account when constructing experimental designs to measure these effects.

II-1. Three treatments with two orthogonal 3×3 latin squares.

For $t = 3$ treatments in $t - 1 = 2$ orthogonal 3×3 latin squares the following design is one which tends to equalize the relative effective number of replicates for direct and residual effects as the number of rows increases:

Period or row number	Sequence or column number				
	s = 1	s = 2	s = 3	s = 4	...
	1 2 3	4 5 6	7 8 9	10 11 12	13
q = 1 3 + 1 = 4	1 A B C	A B C	A B C	A B C	A
	2 B C A	C A B	B C A	C A B	B
	3 C A B	B C A	C A B	B C A	C
	4 A B C	A B C	A B C	A B C	A
q = 2 3(2) + 1 = 7	5 C A B	B C A	C A B	B C A	C
	6 B C A	C A B	B C A	C A B	B
	7 A B C	A B C	A B C	A B C	A
q = 3 3(3) + 1 = 10	8 B C A	C A B	B C A	C A B	B
	9 C A B	B C A	C A B	B C A	C
	10 A B C	A B C	A B C	A B C	A
q = 4 3(4) + 1 = 13	11 C A B	B C A	C A B	B C A	C
	12 B C A	C A B	B C A	C A B	B
	13 A B C	A B C	A B C	A B C	A
14 ⋮	B C A	C A B	B C A	C A B	B

In the above, the three columns for s an even number are identical; likewise, the three columns for s an odd number are identical. In a similar manner, rows 2, 3, and 4 are repeated for q an odd number and rows 5, 6, and 7 are repeated

for q an even number. Also, the first three rows of the first six columns is an ordinary double change-over design made up of the two orthogonal 3×3 latin squares.

A second design could be obtained by repeating rows 2, 3, and 4 (or alternatively, rows 5, 6, and 7) for all q . For this design $s \geq 2$ is required, and in order to attain the efficiency of the previous design s must be an even number.

Other designs are possible by making use of repetition both vertically (over rows) and horizontally (over columns) and by having treatments follow themselves (e.g. see Federer [1955], Patterson and Lucas [1959], and Atkinson [1963]).

II-2. Four treatments with three orthogonal 4×4 latin squares.

For $t = 4$ treatments in $(t - 1) = 3$ orthogonal 4×4 latin squares with $r = 4q + 1$ rows and $c = 4s$ columns, the design is:

Row or period number		Sequence or column number																
		s = 1				s = 2				s = 3				s = 4				...
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	
q = 1	1	A	B	C	D	A	B	C	D	A	B	C	D	A	B	C	D	
	2	B	A	D	C	D	C	B	A	C	D	A	B	B	A	D	C	
	3	C	D	A	B	B	A	D	C	D	C	B	A	C	D	A	B	
	4	D	C	B	A	C	D	A	B	B	A	D	C	D	C	B	A	
4 + 1 =	5	A	B	C	D	A	B	C	D	A	B	C	D	A	B	C	D	
q = 2	6	C	D	A	B	B	A	D	C	D	C	B	A	C	D	A	B	
	7	D	C	B	A	C	D	A	B	B	A	D	C	D	C	B	A	
	8	B	A	D	C	D	C	B	A	C	D	A	B	B	A	D	C	
	4(2) + 1 =	9	A	B	C	D	A	B	C	D	A	B	C	D	A	B	C	
q = 3	10	D	C	B	A	C	D	A	B	B	A	D	C	D	C	B	A	
	11	B	A	D	C	D	C	B	A	C	D	A	B	B	A	D	C	
	12	C	D	A	B	B	A	D	C	D	C	B	A	C	D	A	B	
	4(3) + 1 =	13	A	B	C	D	A	B	C	D	A	B	C	D	A	B	C	
q = 4	14	B	A	D	C	D	C	B	A	C	D	A	B	B	A	D	C	.
	15	C	D	A	B	B	A	D	C	D	C	B	A	C	D	A	B	
	16	D	C	B	A	C	D	A	B	B	A	D	C	D	C	B	A	
	4(4) + 1 =	17	A	B	C	D	A	B	C	D	A	B	C	D	A	B	C	
⋮																		

In the above design the four columns or sequences for $s = 1, 4, 7, \dots$ are identical; the four columns for $s = 2, 5, 8, \dots$ are identical; and the four columns for $s = 3, 6, 9, \dots$ are also identical. Likewise, rows 2 to 5 are

identical to rows 14 to 17, 26 to 29, ...; rows 6 to 9 are identical to rows 18 to 21, 30 to 33, ...; and rows 10 to 13 are identical to rows 22 to 25, 34 to 37, This repetitive scheme horizontally and vertically continues for s and q any positive integer.

As for $t = 3$, alternative designs for $t = 4$ are available utilizing one or two squares and repetitions of these. Also, designs from the first one described above are available for r and c any positive integer but considerable balance is obtained when $r = 4q + 1$ and $c = 4s$, resulting in a simplified analysis.

II-3. Use of one or more squares for t an even number.

It has been demonstrated by Williams [1949] and Ferris [1957] that one square is sufficient to estimate residual and direct effects when t is an even number. They have listed squares for this purpose. However, if more than $t + 1$ rows and t columns are to be used, a more efficient design in the sense of equalizing the variances on residual and direct effects may be obtained by taking the additional rows and columns from additional orthogonal latin squares (e.g. see Bose et al. [1960, 1960, 1961] for a discussion on construction and existence of orthogonal latin squares and see Fisher and Yates [1938] for a listing of orthogonal squares for t a prime number or power of a prime number).

The one or more orthogonal basic squares used in a design could be repeated both vertically and horizontally to obtain the desired number of rows and columns. But, as before, we note that this type of design will not be as efficient as using all $t - 1$ orthogonal latin squares, or as many orthogonal squares as exist, to construct the design.

II-4. Use of two or more squares for t odd.

Williams [1949] and Ferris [1957] have pointed out that a minimum of two orthogonal latin squares is necessary to achieve any type of balance and still allow estimation of direct and residual effects when t is an odd number. (For $t = 3$, say, one could estimate residual and direct effects from 3 rows and 5 columns but this would be an unbalanced arrangement.) The use of as many orthogonal latin squares as possible would be preferred for any such repetitive scheme as described in sections II-1 and II-2 in order to improve the efficiency of an experimental design.

III. RANDOMIZATION

Once the design has been determined the sequence of treatments in any column is fixed. Therefore, there can be no randomization among the rows (periods); the randomization is confined to randomly allotting the columns to the experimental units and the letters to the treatments. If there is no stratification among the experimental units receiving the columns, the following randomization procedure suffices:

- i) Randomly assign the t treatments to the t letters A, B, C, D,
- ii) Randomly assign the c experimental units (i.e., stores, patients, students, cows, rats, machines, etc.) receiving a sequence of treatments to the c sequences (columns) .

If the columns are stratified into categories or sets of size t each, then the second step above is altered as follows:

- ii)' Randomly assign the t units in each category to the t columns for $s = 1, s = 2$, etc. using a different randomization for each s .

If the columns are stratified into categories of sizes not equal to t then either ii) or ii)' may be followed unless there is a category by residual effect or/and category by direct effect interaction. In this event, randomize the required number of sequences within each category. An unbalanced feature is introduced into the analysis thus complicating the arithmetic and the algebra. ^A

^A Appropriate designs for efficiently measuring residual and direct treatment effects and interactions are not considered in the present paper.

IV. ESTIMATORS FOR EFFECTS

Suppose that the observational yield Y_{ijh} is such that it is expressible in the following linear additive form:

$$Y_{ijh} = N_{ijh}(\mu + \gamma_i + \beta_j + \delta_h + \sum_{p=1}^t N_{i(j-1)p} \rho_p + \epsilon_{ijh}), \quad (1)$$

where $N_{ijh} = 1$ if the h^{th} treatment appears in i^{th} column and j^{th} row and equals zero otherwise, $N_{i(j-1)p} = 1$ if the p^{th} treatment appeared in row $j-1$ and in the i^{th} column and equals zero otherwise, μ is an effect common to all observations, γ_i = effect of i^{th} column, β_j = effect of j^{th} row, δ_h = direct effect of h^{th} treatment, ρ_p = residual effect of p^{th} treatment in the period (row) immediately following the period (row) in which the treatment was applied, ϵ_{ijh} are identically independently distributed random variates with mean zero and common variance σ_ϵ^2 , $i = 1, 2, \dots, c$, $j = 1, 2, \dots, r$, $h = 1, 2, \dots, t$, and $p = 1, 2, \dots, t$.

In general N_{ijh} could be the number of observations for the h^{th} treatment in the j^{th} row and i^{th} column rather than only taking on the values of zero and one. However, this generality is not followed for the analyses below. Also, $\sum_{h=1}^t N_{ijh}$ could be zero for certain i and j instead of unity for the results in the following section. However, for the results in subsections IV-2 and IV-3, and in section V it is assumed that $\sum_{h=1}^t N_{ijh} = 1$. This makes row and column effects orthogonal.

IV-1. r and c any positive integer.

Differentiation of the residual sum of squares with respect to the parameters and equating the resulting equations to zero result in the following normal equations:

For μ :

$$N \dots \mu + \sum_{i=1}^c N_{i..} \gamma_i + \sum_{j=1}^r N_{.j.} \beta_j + \sum_{h=1}^t N_{..h} \delta_h + \sum_{i=1}^c \sum_{j=1}^r \sum_{h=1}^t \sum_{p=1}^t N_{ijh} N_{i(j-1)p} \rho_p$$

$$= Y \dots = \text{grand total} \quad (2)$$

For γ_i :

$$N_{i..} (\mu + \gamma_i) + \sum_{j=1}^r N_{ij.} \beta_j + \sum_{h=1}^t N_{i.h} \delta_h + \sum_{j=1}^r \sum_{h=1}^t \sum_{p=1}^t N_{ijh} N_{i(j-1)h} \rho_p$$

$$= Y_{i..} = i^{\text{th}} \text{ column total} \quad (3)$$

For β_j :

$$N_{.j.} (\mu + \beta_j) + \sum_{i=1}^c N_{ij.} \gamma_i + \sum_{h=1}^t N_{.jh} \delta_h + \sum_{i=1}^c \sum_{h=1}^t \sum_{p=1}^t N_{ijh} N_{i(j-1)p} \rho_p$$

$$= Y_{.j.} = j^{\text{th}} \text{ row total} \quad (4)$$

For δ_h :

$$N_{..h} (\mu + \delta_h) + \sum_{i=1}^c N_{i.h} \gamma_i + \sum_{j=1}^r N_{.jh} \beta_j + \sum_{i=1}^c \sum_{j=1}^r \sum_{p=1}^t N_{ijh} N_{i(j-1)p} \rho_p$$

$$= Y_{..h} = h^{\text{th}} \text{ treatment total} \quad (5)$$

For ρ_p :

$$\sum_{i=1}^c \sum_{j=1}^r \sum_{h=1}^t N_{ijh} N_{i(j-1)p} (\mu + \gamma_i + \beta_j + \delta_h + \rho_p)$$

$$= \sum_{i=1}^c \sum_{j=1}^r \sum_{h=1}^t N_{ijh} N_{i(j-1)p} Y_{ijh} = \text{total}$$

for all observations receiving the p^{th} treatment in the preceding period (row) = Y_{pp}

(6)

A unique solution for the parameters is obtained with the addition of an appropriate set of restraints, e.g. the following,

$$\sum_{i=1}^c \gamma_i = \sum_{j=1}^r \beta_j = \sum_{h=1}^t \delta_h = \sum_{p=1}^t \rho_p = 0 \quad , \quad (7)$$

and for certain minimum values of r and c . For example, if complete squares are used then either $r \geq 2t + 1$ and/or $c \geq 2t$ for t , r , and c any positive integer.

Although equations (2) to (4) could be used to substitute in equations (5) and (6) to obtain equations involving only δ_h and ρ_p parameters this was not done because particular solutions will be obtained for special values of r and c . Also, this form of the normal equations is more appropriate for the various forms that tied-double-change-over designs can take. If $\sum_{h=1}^t N_{ijh} = 0$ for some i and j these equations are immediately useful for these cases, as well as for N_{ijh} equal any integer.

IV-2. $r = tq + 1$, $c = ts$, and $sq = k(t-1)$ for k a positive integer.

When $r = tq + 1$, $c = ts$, $sq = k(t-1)$, k a positive integer, $\sum_{h=1}^t N_{ijh} = 1$ for all ij , and with the restraints in equation (7) normal equations (2) to (6) reduce to:

$$st(tq + 1)\hat{\mu} = N \dots \hat{\mu} = Y \dots \quad (8)$$

$$(tq + 1)(\hat{\mu} + \hat{\gamma}_1) + \bar{\epsilon}_f = Y_{1..} \quad (9)$$

(where $f = \text{remainder of fraction } 1/t$, not a positive integer, and for $1/t$ a positive integer $f = t$ when the order of the design is as described in sections II-1 and II-2.)

$$ts(\hat{\mu} + \hat{\beta}_j) = Y_{.j} \quad (10)$$

$$s(tq + 1)(\hat{\mu} + \hat{\delta}_h) + (\hat{\gamma}_h + \hat{\gamma}_{t+h} + \dots + \hat{\gamma}_{t(s-1)+h})$$

$$- \frac{tsq}{t-1} \hat{\rho}_h = Y_{..h} \quad (11)$$

$$stq(\hat{\mu} + \hat{\rho}_p) - s\hat{\beta}_1 - \frac{tsq}{t-1} \hat{\delta}_p = Y_{pp} \quad (12)$$

Substitution of the solutions $\hat{\mu}$, $\hat{\gamma}_1$, and $\hat{\beta}_1$ in equations (11) and (12) results in:

$$s(tq + 1 - \frac{1}{tq+1}) \hat{\delta}_h - \frac{tsq}{t-1} \hat{\rho}_h = Y_{..h} - \sum_{q=1}^s \bar{y}_{[(q-1)t+h]} \dots$$

$$- stq \bar{y} = Q_{..h} \quad (13)$$

$$stq \hat{p}_p - \frac{stq}{t-1} \hat{\delta}_p = Y_{pp} + s\bar{y}_{.1} - s(tq+1)\bar{y} = R_{..p} \quad (14)$$

Let $\hat{\delta} = a t \times 1$ column vector of treatment direct effects, let $\hat{p} = a t \times 1$ column vector of residual treatment effects, let Q be a $t \times 1$ column vector for right hand side of (13), and let R be a $t \times 1$ column vector for the right hand side of (14). Then, the solution for direct and residual effects eliminating all other effects is:

$$\begin{bmatrix} \hat{\delta} \\ \hat{p} \end{bmatrix} = \frac{(t-1)(tq+1)}{stq[t(t-2)(tq+2)+1]} \begin{bmatrix} (t-1)I & I \\ I & \frac{(t-1)(tq+2)}{tq+1}I \end{bmatrix} \begin{bmatrix} Q \\ R \end{bmatrix} \quad (15)$$

where I is a $t \times t$ identity matrix.

Likewise, the solutions for the direct effects ignoring the residual effects (i.e. setting the residual effects equal to zero and then solving for the direct effects) and for the residual effects ignoring the direct effects eliminating all other effects is:

$$\begin{bmatrix} \delta' \\ p' \end{bmatrix} = \frac{1}{stq} \begin{bmatrix} \frac{tq+1}{tq+2}I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} Q \\ R \end{bmatrix} \quad (16)$$

With the results in this form the variances of differences between direct effects and between residual effects is readily obtainable. To illustrate, the solutions for $t = 3$ are:

$$\begin{bmatrix} \hat{\delta}_1 \\ \hat{\delta}_2 \\ \hat{\delta}_3 \\ \hat{p}_1 \\ \hat{p}_2 \\ \hat{p}_3 \end{bmatrix} = \frac{2(3q+1)}{3sq(9q+7)} \begin{bmatrix} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & \frac{2(3q+2)}{3q+1} & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{2(3q+2)}{3q+1} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{2(3q+2)}{3q+1} \end{bmatrix} \begin{bmatrix} Q_{..1} \\ Q_{..2} \\ Q_{..3} \\ R_{..1} \\ R_{..2} \\ R_{..3} \end{bmatrix} \quad (17)$$

The variance of the difference $\hat{\delta}_1 - \hat{\delta}_2$ is

$$V(\hat{\delta}_1 - \hat{\delta}_2) = \sigma_{\epsilon}^2 \frac{2(3q+1)}{3sq(9q+7)} \left\{ 2 + 2 - 0 - 0 \right\} = \frac{8(3q+1)}{3sq(9q+7)} \sigma_{\epsilon}^2 \quad (18)$$

and the variance of a difference between two residual effects, say $\hat{p}_1 - \hat{p}_2$ is

$$V(\hat{p}_1 - \hat{p}_2) = \frac{8(3q+2)}{3sq(9q+7)} \sigma_{\epsilon}^2 \quad (19)$$

The ratio of the variances of direct effects to residual effects is $(3q+1)/(3q+2)$ which rapidly approaches unity as q increases.

For $t = 4$ treatments the solutions are:

$$\begin{bmatrix} \hat{\delta}_1 \\ \hat{\delta}_2 \\ \hat{\delta}_3 \\ \hat{\delta}_4 \\ \hat{p}_1 \\ \hat{p}_2 \\ \hat{p}_3 \\ \hat{p}_4 \end{bmatrix} = \frac{3(4q+1)}{4sq(32q+17)} \begin{bmatrix} 3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & \frac{3(4q+2)}{4q+1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{3(4q+2)}{4q+1} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{3(4q+2)}{4q+1} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{3(4q+2)}{4q+1} \end{bmatrix} \begin{bmatrix} Q_{..1} \\ Q_{..2} \\ Q_{..3} \\ Q_{..4} \\ R_{..1} \\ R_{..2} \\ R_{..3} \\ R_{..4} \end{bmatrix} \quad (20)$$

The variance between two estimated direct effects, say $\hat{\delta}_1$ and $\hat{\delta}_2$ is

$$V(\hat{\delta}_1 - \hat{\delta}_2) = \frac{9(4q+1)}{2sq(32q+17)} \sigma_\epsilon^2 \quad (21)$$

and the variance between two estimated residual effects, say \hat{p}_1 and \hat{p}_2 is

$$V(\hat{p}_1 - \hat{p}_2) = \frac{9(4q+2)}{2sq(32q+17)} \sigma_\epsilon^2 \quad (22)$$

The ratio of variances of estimated direct effects to residual effects is $(4q+1)/(4q+2)$.

IV-3. Conditions of section IV-2 with first row omitted.

In certain situations it may be undesirable to include the results of the first period due to residual effects from the periods preceding the start of the experiment. The omission of results from period (row) one can be handled very simply. Observing the form of equation (12) and the form of $R_{..p}$ in equation (14), the omission of first row data deletes the $-s\hat{\beta}_1$ term from equation (12) and changes $R_{..p}$ into the following form:

$$R'_{..p} = Y_{\rho p} - stq \bar{y} \quad . \quad (23)$$

Now simply replace $R_{..p}$ with $R'_{..p}$ in equations (15) and (16) and the solutions are obtained in the same manner as before.

V. SUMS OF SQUARES

The analysis of variance for the tied-double-change-over design takes on the following form for $\sum_{h=1}^t N_{ijh} = 1$ for all ij , for $r = tq + 1$, and $c = ts$:

Source of variation	df	Sum of squares
Total	rc	$\sum_{i=1}^c \sum_{j=1}^r Y_{ijh}^2$
Correction for mean	1	$Y_{...}^2 / rc$
Columns (ign. direct effects)	c-1	$\sum_{i=1}^c Y_{i..}^2 / r - Y_{...}^2 / rc$
Rows	r-1	$\sum_{j=1}^r Y_{.j.}^2 / c - Y_{...}^2 / rc$
Col. x row interaction (ign. direct and residual effects)	(r-1)(c-1)	$\sum_{i=1}^c \sum_{j=1}^r Y_{ijh}^2 - \sum_{i=1}^c Y_{i..}^2 / r - \sum_{j=1}^r Y_{.j.}^2 / c + Y_{...}^2 / rc$
Direct effects (eliminating rows and cols.; ignoring residual effects)	t-1	$\sum_{h=1}^t \delta'_h Q_{..h}$
Residual effects (eliminating all other effects)	t-1	$\sum_{h=1}^t \hat{\delta}_h Q_{..h} + \sum_{p=1}^t \hat{p}_{..p} R_{..p} - \sum_{h=1}^t \delta'_h Q_{..h}$
Error	(r-1)(c-1) -2(t-1)	by subtraction
Direct effects (elim. all other effects)*	t-1	$\sum_{h=1}^t \hat{\delta}_{..h} Q_{..h} + \sum_{p=1}^t (\hat{p}_p - \rho'_p) R_{..p}$
Residual effects (ign. direct effects; elim. all other)	t-1	$\sum_{p=1}^t \rho'_p R_{..p}$

The symbols in the above analysis of variance are defined in the preceding equations. The sums of squares for direct effects (eliminating all other effects) and for residual effects (eliminating all other effects) may be slightly simplified by obtaining solutions from (13) and (14) for direct effects and residual effects, respectively. The $Q_{..h}$ and the $R_{..p}$ values will be changed.

The "Error" sum of squares divided by the degrees of freedom, $(r-1)(c-1) - 2(t-1)$, yields an estimate of σ_{ϵ}^2 which is used in equations (18), (19), (21), and (22) to obtain the estimated variance of differences between effects.

The above form for the analysis of variance holds for any r and c and/or for $\sum_{h=1}^t N_{ijh}$ equal zero or one. The sums of squares in the top part of the table will be columns ignoring all other effects but the mean, rows eliminating columns and mean but ignoring all other effects, and column \times row interaction eliminating rows, columns, and mean but ignoring direct and residual treatment effects. The last sum of squares is partitioned in the same manner as described in the last two portions of the above table. The solutions for \hat{p}_p , $\hat{\delta}_h$, ρ'_p , and δ'_h are obtained from the equations in subsection III-1, i.e., (2) to (7).

VI. A NUMERICAL EXAMPLE

The following example with artificial data is used to illustrate the numerical computations for a tied-double-change-over design:

Row	Column						Total
	1	2	3	4	5	6	
1	C 21	C 21	A 11	A 12	B 7	B 18	90
2	A 16	B 20	C 25	B 16	C 20	A 23	120
3	B 18	A 15	B 19	C 26	A 11	C 31	120
4	C 26	C 25	A 16	A 17	B 11	B 25	120
5	B 19	A 19	B 18	C 26	A 11	C 33	126
6	A 19	B 18	C 28	B 22	C 21	A 24	132
7	C 27	C 28	A 19	A 17	B 15	B 26	132
Totals	146	146	136	136	96	180	840

The various other totals required are (see equations (13) and (14)):

$$\begin{array}{ll}
 Y_{..A} = 230 & Q_{..A} = 230 - (136 + 136)/7 - 240 = -342/7 \\
 Y_{..B} = 252 & Q_{..B} = 252 - (96 + 180)/7 - 240 = -192/7 \\
 Y_{..C} = 358 & Q_{..C} = 358 - (146 + 146)/7 - 240 = 534/7 \\
 Y_{pA} = 262 & R_{..A} = 262 + 30 - 280 = 12 \\
 Y_{pB} = 262 & R_{..B} = 262 + 30 - 280 = 12 \\
 Y_{pC} = 226 & R_{..C} = 226 + 30 - 280 = -24
 \end{array}$$

From equation (17) the solutions for the effects are:

$$\begin{bmatrix} \hat{\delta}_A \\ \hat{\delta}_B \\ \hat{\delta}_C \\ \hat{p}_A \\ \hat{p}_B \\ \hat{p}_C \end{bmatrix} = \frac{7}{150} \begin{bmatrix} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 16/7 & 0 & 0 \\ 0 & 1 & 0 & 0 & 16/7 & 0 \\ 0 & 0 & 1 & 0 & 0 & 16/7 \end{bmatrix} \cdot \begin{bmatrix} -342/7 \\ -192/7 \\ 534/7 \\ 12 \\ 12 \\ -24 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ 6 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

From equation (16) the solutions for direct ignoring residual and residual ignoring direct effects are:

$$\begin{bmatrix} \delta'_A \\ \delta'_B \\ \delta'_C \\ p'_A \\ p'_B \\ p'_C \end{bmatrix} = \frac{1}{12} \begin{bmatrix} 7/8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 7/8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 7/8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -342/7 \\ -192/7 \\ 534/7 \\ 12 \\ 12 \\ -24 \end{bmatrix} = \begin{bmatrix} -57/16 \\ -32/16 \\ 89/16 \\ 1 \\ 1 \\ -2 \end{bmatrix}$$

Utilizing the above results, the sums of squares and mean squares in the analysis of variances follow directly as:

Source of variation	d.f.	Sum of squares	Mean square
Total	42	18,212	—
Correction for mean	1	16,800	—
Columns (ignoring direct effects)	5	520	—
Rows	6	204	—
Column x row interaction (ignoring direct and residual effects)	30	688	—

Direct (ign. residual; eliminating all other effects)	2	653.25	—
Residual (eliminating all other effects)	2	18.75	75/8
Error	26	16.00	8/13

Direct (eliminating all other effects)	2	624.00	312
Residual (ignoring direct; eliminating all other effects)	2	48.00	—

The column x row interaction is partitioned into the direct plus residual sum of squares, i. e.

$$\{(-4)(-342) + (-2)(-192) + 6(534)\} / 7 - 12 - 24 = 672 ,$$

and the error sum of squares, $688 - 672 = 16.00$.

The sums of squares for direct ignoring residual and eliminating all else and residual ignoring direct and eliminating all else are

$$\{(-57)(-342) + (-32)(-192) + 89(534)\} / 7(16) = 653.25 \text{ and } 1(12) + 1(12) + (-2)(-24) = 48 ,$$

respectively. Therefore, the sums of squares for direct effects eliminating all else and for residual effects eliminating all else are

$$672 - 48 = 624 \text{ and } 672 - 653.25 = 18.75 ,$$

respectively.

From equations (18) and (19) the estimated variances of differences of direct and of residual effects are

$$\frac{8(7)}{12(25)} \left(\frac{8}{13} \right) = 0.1149 \quad \text{and} \quad \frac{8(8)}{12(25)} \left(\frac{8}{13} \right) = 0.1313 .$$

The ratio of the variances for direct effects to residual effects is $7/8$.

If variances of differences of direct plus residual equal permanent effects are required these are readily obtained from the results in equations (15) and (17) . For example the difference between the estimated permanent effects for treatment A and B is $\hat{\delta}_A + \hat{\rho}_A - \hat{\delta}_B - \hat{\rho}_B = -4 - 1 - (-2) - 0 = -3$; the estimated variance of this difference is

$$\frac{2(8)}{13} \left(\frac{7}{150} \right) \left(2 + 16/7 + 2(1) \right) = 128/325 = .3938 .$$

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